Use of Multicore Architectures in Solving Time Dependent Dynamical Systems

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Plan of Presentation

1. Rescaling Dynamical Systems
2. Motion of a membrane linked to a spring
3. Reaction-diffusion evolution problem
4. Satellite trajectories obeying non-Keplerian laws

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Rescaling Dynamical Systems —> Generation of a Coarse Grid of Time Slices (Multiple Shooting method)

System of ODE's:

$$(S) \qquad \begin{cases} \frac{dY}{dt} = F(Y), & 0 < t \leqslant T \leqslant \infty, \\ Y(0) = Y_0. \end{cases}$$

Equivalent sequence of Initial Value Shooting Problems:

$$(S_n) \qquad \begin{cases} \frac{dY}{dt} = F(Y), & T_{n-1} < t \leqslant T_n \\ Y(T_{n-1}) = Y_{n-1}, & \\ \text{EOS condition for getting } T_n. & \end{cases}$$

Resulting coarse grid $\{T_n\}$ such that: $\cup_{n \geqslant 1}[T_{n-1}, T_n] = [0, T]_{T \leqslant \infty}$.

Preliminary Rescaling Technique

Change of variables: $\{t, Y\} \longrightarrow \{s, Z\}$

$$\begin{cases} t = T_{n-1} + \beta_n s, \quad \beta_n > 0 \\ Y(t) = Y_{n-1} + D_n Z(s), \end{cases}$$

Equivalent sequence of rescaled Initial Value Shooting Problems:

$$(S'_n) \quad \begin{cases} \frac{dZ}{ds} = G_n(Z(s)), & T_{n-1} < t \leqslant T_n \\ Z(0) = 0, & \\ \text{Rescaled EOS condition for getting } s_n. & \end{cases}$$

Time Parallel Algorithm (RaPTI)

Step 1- **Choice of a uniform stopping criterion**

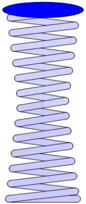
Step 2- **Prediction of the starting values of the solution using the Ratio Method**

Step 3- **Iteration until convergence of all slices of:**

- **Parallel computation**, on fine grids, of independent IVPs
- **Evaluation of the “gaps"** between the end value of each slice and the predicted starting value of the next slice.
- **Correction procedure**, updating the predictions of the starting values of the solution using the **Ratio Method** at the onset of each of the not-yet-converged time-slices.

Membrane Problem: A Benchmark Model for Computation

$$\begin{cases} y'' - b|y'|^{q-1}\, y' \; + \; |y|^{p-1}\, y = 0, \;\; t > 0, \\ y(0) = y_{1,0}, \\ y'(0) = y_{2,0}. \end{cases} \tag{1}$$



Behavior of the Solution

We consider the case: $p \leqslant q \leqslant \frac{2p}{p+1}$ for which the corresponding blow-up time is infinite with both y and y' exhibiting oscillatory behaviors:

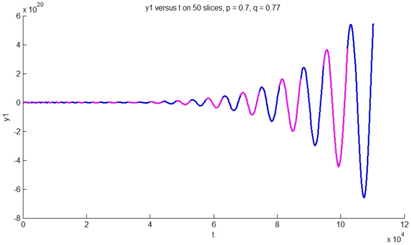


Figure: Oscillatory Behavior of y

Lowering the Order

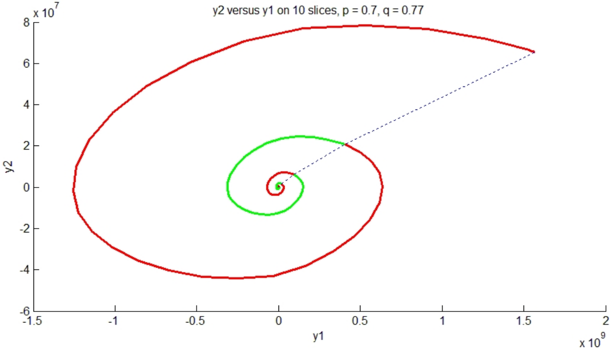
Change of variables:

$$Y_1 = y, \quad Y_2 = y'.$$

This would yield:

$$\begin{cases} Y_1'(t) = Y_2, \\ Y_2'(t) = b|Y_2|^{q-1}\, Y_2 \; - \; |Y_1|^{p-1}\, Y_1, \\ Y_1(0) = Y_{1,0}, \\ Y_2(0) = Y_{2,0}. \end{cases} \tag{2}$$

End-of-Slice Condition



$$(Y_{i,2})^2 = |Y_{i,1}|^{p+1}, \quad Y_{i,1} > 0, \quad Y_{i,2} > 0 \tag{3}$$

Speed-up and Efficiency

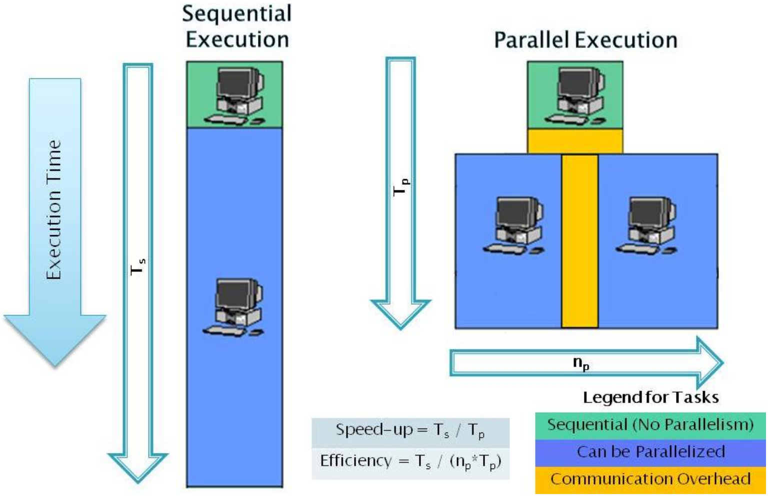
$$\text{Speed-up} = \frac{T_S}{T_P}$$

$$\text{Efficiency} = \frac{T_S}{n_P \times T_P}$$

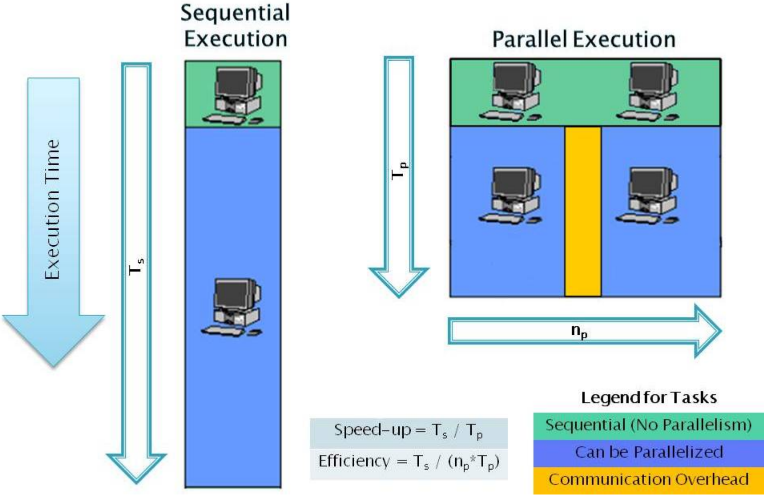
Amdahl's Law: The speed-up is limited by the time needed for the sequential fraction of the program.

$$\boxed{S_{n_p}^{max} = \frac{1}{(1-P)+\frac{P}{n_p}}.}$$

Classical Approach



Duplication Approach



Numerical Results: $p \leqslant q < \frac{2p}{p+1}$

Case	1	2	3	4	5	6	7
ϵ_{tol}^{g}	$5*10^{-6}$	10^{-5}	10^{-4}	$5*10^{-6}$	10^{-5}	$5*10^{-6}$	10^{-5}
ϵ_{tol}^{r}	10^{-5}	10^{-4}	10^{-4}	10^{-5}	10^{-4}	10^{-5}	10^{-4}
p	0.8	0.8	0.8	0.7	0.7	0.7	0.7
q	0.84	0.84	0.84	0.735	0.735	0.77	0.77
N	**65000**	**65000**	**65000**	**65000**	**65000**	**50000**	**50000**
$\mathbf{n_s}$	1499	469	469	1143	361	1471	462
P	97.69%	99.28%	99.28%	98.24%	99.44%	97.06%	99.08%
$\mathbf{n_I}$	**6**	**6**	**2**	**11**	**14**	**12**	**16**
T_s	252.81	252.81	252.81	255.5	255.5	199.94	199.94
T_p^2	134.81	130.55	130.26	132.27	130.08	103.37	101.42
E_2	0.938	0.968	0.970	0.966	0.982	0.967	0.986
$\mathbf{S_2}$	**1.875**	**1.936**	**1.941**	**1.932**	**1.964**	**1.934**	**1.971**
S_2^{max}	1.955	1.986	1.986	1.965	1.989	1.943	1.982
T_p^4	70.91	67.06	67.49	69.74	67.68	57.11	53.73
E_4	0.891	0.942	0.936	0.916	0.944	0.875	0.930
$\mathbf{S_4}$	**3.565**	**3.770**	**3.746**	**3.664**	**3.775**	**3.501**	**3.721**
S_4^{max}	3.741	3.915	3.915	3.800	3.934	3.676	3.892
T_p^8	39.07	35.48	35.14	37.82	35.77	32.1	28.57
E_8	0.809	0.891	0.899	0.844	0.893	0.779	0.875
$\mathbf{S_8}$	**6.471**	**7.125**	**7.194**	**6.756**	**7.143**	**6.229**	**6.998**
S_8^{max}	6.888	7.615	7.615	7.123	7.701	6.634	7.514

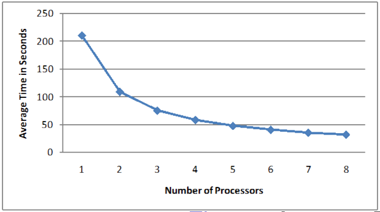
Numerical Results: $p \leqslant q < \frac{2p}{p+1}$

Case	8	9	10	11	12	13	14
ϵ^g_{tol}	$5 * 10^{-6}$	10^{-5}	$5 * 10^{-6}$	10^{-5}	$5 * 10^{-6}$	10^{-5}	$5 * 10^{-6}$
ϵ^r_{tol}	10^{-5}	10^{-4}	10^{-5}	10^{-4}	10^{-5}	10^{-4}	10^{-5}
p	0.7	0.7	0.6	0.6	0.6	0.6	0.6
q	0.805	0.805	0.66	0.66	0.69	0.69	0.72
N	**9000**	**9000**	**65000**	**65000**	**65000**	**65000**	**65000**
n_s	2497	775	1156	365	1414	451	1993
P	72.26%	91.39%	98.22%	99.44%	97.82%	99.31%	96.93%
n_I	**8**	**12**	**35**	**37**	**28**	**29**	**23**
T_s	34.84	34.84	265.9	265.9	262.89	262.89	261.47
T^2_p	23.41	20.15	135.87	134.65	135.66	135.36	136.64
E_2	0.744	0.865	0.979	0.987	0.969	0.971	0.957
S_2	**1.488**	**1.729**	**1.957**	**1.975**	**1.938**	**1.942**	**1.914**
S^{max}_2	1.566	1.841	1.965	1.989	1.957	1.986	1.941
T^4_p	17.76	12.37	74.06	71.82	73.77	71.13	76.06
E_4	0.490	0.704	0.898	0.926	0.891	0.924	0.859
S_4	**1.962**	**2.816**	**3.590**	**3.702**	**3.564**	**3.696**	**3.438**
S^{max}_4	2.183	3.179	3.797	3.934	3.755	3.918	3.663
T^8_p	14.33	8.44	40.47	37.61	41.18	38.12	43.25
E_8	0.304	0.516	0.821	0.884	0.798	0.862	0.756
S_8	**2.431**	**4.128**	**6.570**	**7.070**	**6.384**	**6.896**	**6.046**
S^{max}_8	2.719	4.991	7.114	7.697	6.943	7.629	6.586

Numerical Results: $p \leqslant q < \frac{2p}{p+1}$

Case	15	16	17	18	19	20	21
ϵ_{tol}^{g}	10^{-5}	$5 * 10^{-6}$	10^{-5}	$5 * 10^{-6}$	10^{-5}	$5 * 10^{-6}$	10^{-5}
ϵ_{tol}^{r}	10^{-4}	10^{-5}	10^{-4}	10^{-5}	10^{-4}	10^{-5}	10^{-4}
p	0.6	0.5	0.5	0.5	0.5	0.5	0.5
q	0.72	0.55	0.55	0.6	0.6	0.65	0.65
N	**65000**	**65000**	**65000**	**65000**	**65000**	**9000**	**9000**
n_s	701	1053	323	1385	422	2716	819
P	98.92%	98.38%	99.50%	97.87%	99.35%	69.82%	90.90%
$\mathbf{n_I}$	**34**	**5**	**5**	**5**	**5**	**2**	**3**
T_s	261.47	264.53	264.53	262.67	262.67	34.42	34.42
T_p^2	136.22	135.28	133.48	135.17	132.94	23.94	20.25
E_2	0.960	0.978	0.991	0.972	0.988	0.719	0.850
$\mathbf{S_2}$	**1.919**	**1.955**	**1.982**	**1.943**	**1.976**	**1.438**	**1.700**
S_2^{max}	1.979	1.968	1.990	1.958	1.987	1.536	1.833
T_p^4	71.99	71.95	68.9	72.34	69.69	18.41	12.39
E_4	0.908	0.919	0.960	0.908	0.942	0.467	0.695
$\mathbf{S_4}$	**3.632**	**3.677**	**3.839**	**3.631**	**3.769**	**1.870**	**2.778**
S_4^{max}	3.875	3.815	3.941	3.760	3.924	2.099	3.142
T_p^8	38.82	38.8	35.9	39.88	36.3	14.98	8.1
E_8	0.842	0.852	0.921	0.823	0.905	0.287	0.531
$\mathbf{S_8}$	**6.735**	**6.818**	**7.369**	**6.587**	**7.236**	**2.298**	**4.249**
S_8^{max}	7.438	7.185	7.731	6.962	7.652	2.570	4.887

Numerical Results:



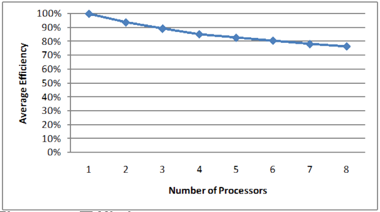


Figure: Average Time & Efficiency

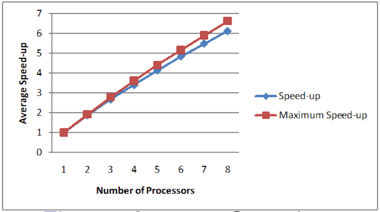


Figure: Average Speed-up

Reaction-Diffusion Problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m = a u^p, \; x \in \Omega \subset \mathbb{R}^d, \; t > 0 & (4.1) \\ u(x,t) = 0, \; x \in \partial\Omega, \; t \geqslant 0, & (4.2) \\ u(x,0) = u_0(x) > 0, \; x \in \Omega. & (4.3) \end{cases} \quad (4)$$

where $a > 0$, $m > 0$, $p > 0$, and Δ is the Laplace operator.

p,m	**Behavior of the Solution**
$0 < p < m$	Bounded behavior and global time existence
$m \leqslant p \leqslant 1$	Global time existence and explosive behavior for certain initial conditions
$m \leqslant 1 < p$	Finite time existence

Table: Behavior of the Solution of the DR Problem

Preliminary Change of Variable

$$\boxed{v = u^m}$$

Let $q = \frac{1}{m}$:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{q}\frac{1}{v^{q-1}}\Delta v + \frac{a}{q}v^{pq-q+1},\ x \in \Omega \subset \mathbb{R}^d,\ t > 0 \\ v(x,t) = 0,\ x \in \partial\Omega,\ t \geqslant 0, \\ v(x,0) = v_0(x) > 0,\ x \in \Omega. \end{cases} \quad (5)$$

with $v_0(x) = [u_0(x)]^m = [u_0(x)]^{\frac{1}{q}}$, and $0 < \frac{1}{q} \leqslant p \leqslant 1$.

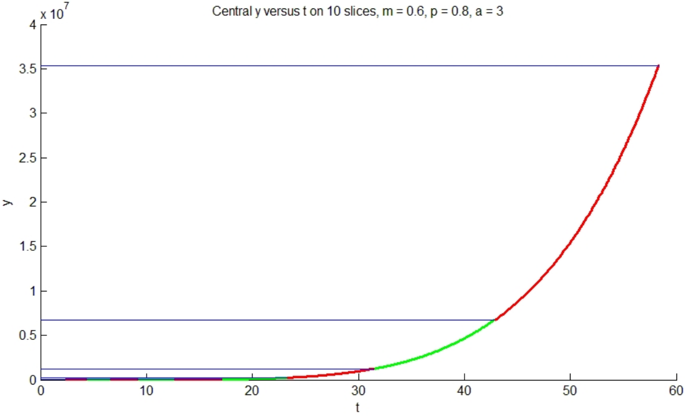
Space Semi-discretization

$A \in \mathbb{R}^{k\times k}$: sparse spd matrix that discretizes the operator $-\Delta$.

$$v(x,t) \approx Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ \ldots \\ Y_k(t) \end{pmatrix} \text{ and } v_0(x) \approx Y_0 = \begin{pmatrix} Y_{0,1} \\ Y_{0,2} \\ \ldots \\ Y_{0,k} \end{pmatrix}$$

$$\begin{cases} \frac{dY}{dt} = -\frac{1}{q} D^{-1}_{Y^{q-1}} AY + \frac{a}{q} Y^{pq-q+1}, \ t > 0 \\ Y(0) = Y_0, \end{cases} \quad (6)$$

End-of-Slice Condition



$$\|Y_i./Y_{i-1}\|_\infty = 1 + S. \quad (7)$$

Numerical Results:

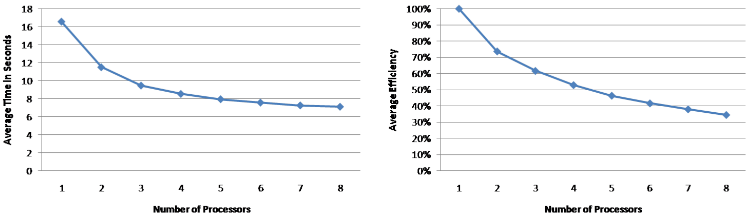


Figure: Average Time & Efficiency

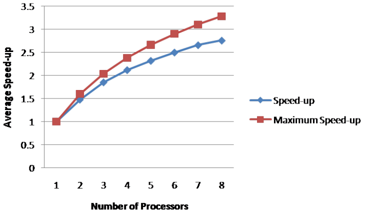


Figure: Average Speed-up

Satellite Problem

General equation of motion given by Newton's second law:

$$\vec{F} = m\ddot{\vec{r}} \tag{8}$$

m mass of the satellite, $\vec{r}$ position-vector, $\vec{F}$ resultant vector-force (forces deriving from gravitational potentials of earth, moon, sun,... and surface forces as atmospheric frictional force, solar radiation pressure,...).

Earth's gravitational potential:

$$U = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n} J_n P_n(\cos\varphi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{a}{r}\right)^{n}\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right] P_{nm}(\cos\lambda)\right\} \tag{9}$$

Satellite Problem

Simplified satellite model, the J_2-model:

$$u(J_2) = \frac{\mu}{r}\left\{1 - \left(\frac{r_{eq}}{r}\right)^2 J_2 P_2(\sin\varphi)\right\} \tag{10}$$

$J_2 = -11.10^{-4}$ is the zonal harmonic of order 2 ($J_1 = 0$),

$P_2(\sin\varphi) = \frac{3}{2}\sin^2\varphi - \frac{1}{2}$ is the second degree Legendre polynomial.

General equation of the J_2-perturbed motion:

$$\ddot{\vec{r}} = \vec{\nabla} u(J_2) \tag{11}$$

Satellite Problem: Differential System

Differential 2^{nd} order system:

$$\begin{cases} \ddot{r}(t) = f(r) = \vec{\nabla} U_{IPQW} = f_K(r) + f_P(r), & (12.1) \\ r(0) = r_0, & (12.2) \\ \dot{r}(0) = \dot{r}_0, & (12.3) \end{cases} \quad (12)$$

Equivalent first-order initial value problem of dimension 6:

$$\begin{cases} \frac{dY}{dt} = F(Y) = F_K(Y) + F_P(Y) \\ Y(0) = Y_0 \end{cases} \quad (13)$$

Satellite Problem: Differential System

$$Y = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad Y_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{pmatrix}, \quad F_K(Y) = \begin{pmatrix} Y_4 \\ Y_5 \\ Y_6 \\ -\mu A \frac{1}{R^3} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \end{pmatrix}$$

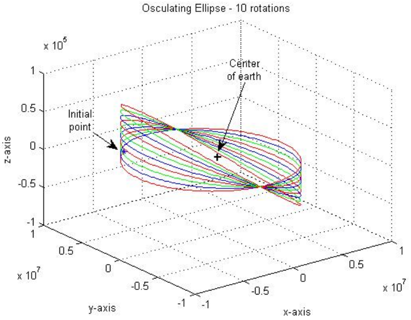
Satellite Problem: Differential System

$$F_P(Y) = \frac{3\mu J_2 r_{eq}^2}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ A\frac{1}{R^5} \begin{pmatrix} Y_1 \\ Y_2 \\ 3Y_3 \end{pmatrix} \end{pmatrix} - \frac{15\mu J_2 r_{eq}^2}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ A\frac{1}{R^7} \begin{pmatrix} Y_3^2 Y_1 \\ Y_3^2 Y_2 \\ Y_3^3 \end{pmatrix} \end{pmatrix}$$

with $R = \sqrt{Y_1^2 + Y_2^2 + Y_3^2}$.

Behavior of the Solution & EOS Condition:

Instantaneous ellipses have the center of earth at one focus.



EOS Condition: $Y_{n,3} = 0$, after completing a whole rotation

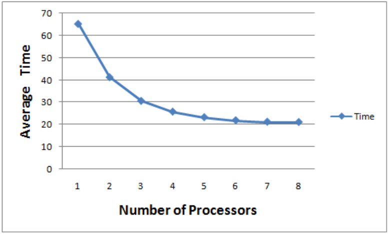
Numerical Results:

Case	1	2	3	4	5	6
e	0.1	0.1	0.1	0.0005	0.15	0.1
a	7300000	7650000	8000000	7300000	7300000	7300000
i_0	98	98	98	98	98	80
ω_0	10	10	10	10	10	10
Ω_0	45	45	45	45	45	45
M_0	123	123	123	123	123	123
N	**1500**	**1500**	**1500**	**1500**	**1500**	**1500**
$\mathbf{n_s}$	**46**	**46**	**46**	**46**	**46**	**46**
P	96.93%	96.93%	96.93%	96.93%	96.93%	96.93%
$\mathbf{n_I}$	**18**	**15**	**14**	**17**	**22**	**48**
T_s	64.17	67.70	70.96	64.35	64.02	64.41
T_p^2	40.48	41.95	44.17	39.67	41.06	43.35
E_2	0.79	0.81	0.80	0.81	0.78	0.74
$\mathbf{S_2}$	**1.59**	**1.61**	**1.61**	**1.62**	**1.56**	**1.49**
S_2^{max}	1.94	1.94	1.94	1.94	1.94	1.94
T_p^4	24.95	25.31	26.36	24.51	26.03	30.20
E_4	0.64	0.67	0.67	0.66	0.61	0.53
$\mathbf{S_4}$	**2.57**	**2.67**	**2.69**	**2.63**	**2.46**	**2.13**
S_4^{max}	3.66	3.66	3.66	3.66	3.66	3.66
T_p^8	19.96	19.45	20.02	19.35	22.29	28.95
E_8	0.40	0.44	0.44	0.42	0.36	0.28
$\mathbf{S_8}$	**3.21**	**3.48**	**3.54**	**3.33**	**2.87**	**2.22**
S_8^{max}	6.59	6.59	6.59	6.59	6.59	6.59

Numerical Results:

Case	8	9	10	11	12	13
e	0.1	0.1	0.1	0.1	0.1	0.1
a	7300000	7300000	7300000	7300000	7300000	7300000
i_0	98	98	98	98	98	98
ω_0	5	20	10	10	10	10
Ω_0	45	45	10	120	45	45
M_0	123	123	123	123	20	60
N	**1500**	**1500**	**1500**	**1500**	**1500**	**1500**
$\mathbf{n_s}$	**46**	**46**	**46**	**46**	**46**	**46**
P	96.93%	96.93%	96.93%	96.93%	96.93%	96.93%
$\mathbf{n_I}$	**34**	**25**	**18**	**18**	**17**	**18**
T_s	65.16	65.06	63.65	64.85	64.81	64.16
T_p^2	42.08	41.00	40.46	40.45	40.38	40.22
E_2	0.77	0.79	0.79	0.80	0.80	0.80
$\mathbf{S_2}$	**1.55**	**1.59**	**1.57**	**1.60**	**1.60**	**1.60**
S_2^{max}	1.94	1.94	1.94	1.94	1.94	1.94
T_p^4	27.25	25.84	24.93	24.78	24.60	24.75
E_4	0.60	0.63	0.64	0.65	0.66	0.65
$\mathbf{S_4}$	**2.39**	**2.52**	**2.55**	**2.62**	**2.63**	**2.59**
S_4^{max}	3.66	3.66	3.66	3.66	3.66	3.66
T_p^8	23.68	20.37	19.96	19.87	19.33	20.21
E_8	0.34	0.40	0.40	0.41	0.42	0.40
$\mathbf{S_8}$	**2.75**	**3.19**	**3.19**	**3.26**	**3.35**	**3.17**
S_8^{max}	6.59	6.59	6.59	6.59	6.59	6.59

Numerical Results:



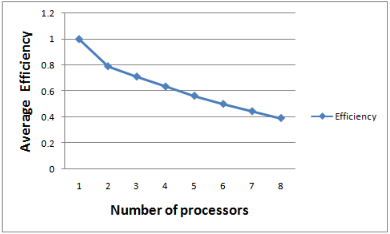


Figure: Average Time & Efficiency

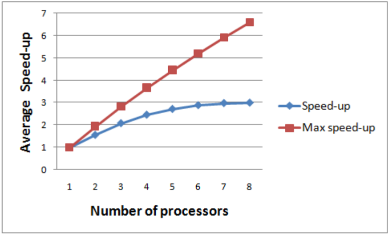


Figure: Average Speed-up

THANK YOU!